SUP Maths Quick Sheet

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Derivatives

$$[f(u(x))]' = f'(u(x)) \times u'(x)$$

$$[e^{u(x)}]' = e^{u(x)} \times u'(x)$$

$$[u(x)^\alpha]' = \alpha u^{\alpha-1}(x) \times u'(x)$$

$$[\ln(u(x))]' = \frac{u'(x)}{u(x)}$$

$$\cos(x)' = -\sin(x)$$

$$\sin(x)' = \cos(x)$$

Primitives

$$u'e^u \to e^u$$

$$u^\alpha u' \to \frac{u^{\alpha+1}}{\alpha+1}$$

$$\frac{u'}{u} \to \ln(u(x))$$

$$u' \sin(u) \to -\cos(u)$$

$$u' \cos(u) \to \sin(u)$$

Taylor Expansions in 0

Exponential:

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + o(x^4)$$

$$(1+x)^\alpha = 1 + \alpha x + \frac{\alpha(\alpha-1)}{2!}x^2 + \frac{\alpha(\alpha-1)(\alpha-2)}{3!}x^3 + o(x^3)$$

$$\frac{1}{1+x} = (1+x)^{-1} = 1 - x + x^2 - x^3 + x^4 + o(x^4)$$

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + x^4 + o(x^4)$$

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + o(x^4)$$

$$\cos(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \left| \begin{matrix} o(x^4) \\ o(x^5) \end{matrix} \right.$$

$$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} + \left| \begin{matrix} o(x^5) \\ o(x^6) \end{matrix} \right.$$

General formula for $f(x)$ when $x \to a$

$$f(x) = f(a) + \frac{f'(a)}{1!}(x-a)^1 + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n$$

Manipulation of $o(x^\alpha)$

- $o(x^\alpha) - o(x^\alpha) = o(x^\alpha)$
- $o(x^\alpha) - o(x^{\alpha+1}) = o(x^\alpha)$
- $o(\lambda x^\alpha) = o(x^\alpha)$
- $x^n \times o(x^\alpha) = o(x^n \times x^\alpha) = o(x^{\alpha+n})$
- $\frac{1}{x^n} o(x^2) = o\left(\frac{1}{x^n} \times x^2\right) = o(x^{2-n})$

Polynomials

Let r be a root of P

- r is a root of order of multiplicity at least m iff
$$(X-r)^m \mid P$$
- r is a root of order of multiplicity exactly m iff
$$\begin{cases}(X-r)^m & \mid P \\ (X-r)^{m+1} & \nmid P\end{cases}$$
- r is a root of order of multiplicity at least m iff
$$\left|\begin{array}{r} P(r)=0 \\ P'(r)=0 \\ \vdots \\ P^{(m-1)}(r)=0 \end{array}\right. \quad m \text{ conditions}$$
- r is a root of order of multiplicity exactly m iff
$$\begin{cases}(X-r)^m & \mid P \\ (X-r)^{m+1} & \nmid P\end{cases}$$

Differential equations

Don't question these formulas. They Just Work™.

First order

With $ay' + by = c$

$$y_0 = ke^{-\int \frac{b}{a}}$$

$$y_p = y_0 \int \frac{c}{ay_0}$$

$$y = y_0 + y_p$$

Second order (constant terms for a b c)

With $ay'' + by' + cy = d(t)$

1. Compute root(s) of $ar^2 + br + c$
   a. $\Delta > 0$: Two real roots r_1 and r_2
      $$y_0 = k_1 e^{r_1 t} + k_2 e^{r_2 t}$$
   b. $\Delta = 0$: One real root r_1
      $$y_0 = (k_1 + k_2 t)e^{r_1 t}$$
   c. $\Delta < 0$: Two complex roots $r_1 = \alpha i + \beta$ and $r_2 = \alpha i - \beta$
      $$y_0 = e^{\alpha t}(k_1 \cos(\beta t) + k_2 \sin(\beta t))$$
2. Getting y_p
   a. $d(t) = P(t)$ (polynomial)
      Then $y_p = Q(t)$
      $$c \neq 0 \quad \rightarrow \deg(Q) = \deg(P)$$
      $$c = 0,\ b \neq 0 \rightarrow \deg(Q) = \deg(P) + 1$$
      $$c = 0,\ b = 0 \rightarrow \deg(Q) = \deg(P) + 2$$
      We then know the expression of $Q(t)$.
      Compute the expressions of $Q'(t)$ and $Q''(t)$.
      $$aQ''(t) + bQ'(t) + cQ(t) = d(t)$$
      Use the coefficients of $d(t)$ to deduce the coefficients of the left side of the equation
   b. $d(t) = P(t)e^{mt}$ (polynomial times exponential)
      Then $y_p = Q(t)e^{mt}$
      Derive y_p twice to get y_p' and y_p''
      $$ay_p'' + by_p' + cy = P(t)e^{mt}$$
      Factorize the left side by e^{mt} and divide both sides by e^{mt}.
      You should find an equation
      $$\alpha Q''(t) + \beta Q'(t) + \gamma Q(t) = P(t)$$
      Once you get this, find $Q(t)$ using the previous method ($d(t) = P(t)$).
   c. For any other kind of $d(t)$
      I'll quote Mehdi for this one:
      "You either Taylor the shit out of it and try to solve for a polynomial, or send it back to the hell it comes from because it won't be on MCQ anyway"
3. $y = y_0 + y_p$

Vector spaces

Direct sum/Supplementary subspaces

$E = F \oplus G$ if both conditions are true:

- $F \cap G = \{0_E\}$
- $F + G = E$
  - $\forall w \in E,\ \exists u \in F,\ \exists v \in G,\ w = u + v$

Linear (in)dependence

A set $X = (x_1, \cdots, x_n) \in E^n$ is linearly independent if

$$\forall (\lambda_i)_{i \in [\![1,n]\!]} \in \mathbb{K}^n, \left(\sum_{i=1}^{n} \lambda_i x_i = 0 \Rightarrow \forall i, \lambda_i = 0 \right)$$

If it is not linearly independent, it is linearly dependent.

- Adding vectors to a linearly dependent set makes it remain dependent.
- Removing vectors from (i.e. taking a subset of) a linearly independent set makes it remain independent.

Span(X)

Let E be a $\mathbb{R}$ vector space.

$$X = \{u_1, u_2, \dots, u_n\} \subset E$$

$$\begin{aligned} Span(X) &= \{\lambda_1 u_1 + \cdots + \lambda_n u_n \mid (\lambda_1, \cdots, \lambda_n) \in \mathbb{R}^{\mathbb{N}}\} \\ &= \{\text{linear combinations of } u_1, \cdots, u_n\} \end{aligned}$$

$Span(X)$ is a $\mathbb{R}$-vs with $X \subset Span(X)$

Spanning set

Let $X \subset E$. We say that X is a spanning set of E if $E = Span(X)$

Basis

A linearly independent spanning set of E is called a basis of E.

$(e_1, \dots, e_n)$ is a basis of E $\Leftrightarrow$ $\forall u \in E$, there exists a unique decomposition of u as a linear combination of the basis $\left(u = \sum_{(i=1)}^{n} \lambda_i e_i\right)$

$$\forall u \in E, \exists! \, (\lambda_1, \dots, \lambda_n) \in \mathbb{R}^n, u = \lambda_1 e_1 + \cdots + \lambda_n e_n$$

Let $\dim(E) = n$, $B = \left(e_1, \dots, e_p\right)$ family of E

Then

- $p < n$, B cannot be a spanning set
- $p > n$, B cannot be independent
- $p = n$, spanning set $\Leftrightarrow$ independent

Linear maps

E and F two $\mathbb{R}$-vs.

$$f : E \to F$$

Then f is a linear map if $\forall (u, v) \in E^2, \forall \lambda \in \mathbb{R}$,

- $f(\lambda u + v) = \lambda f(u) + f(v)$

Or

- $f(u + v) = f(u) + f(v)$
  And $f(\lambda u) = \lambda f(u)$

Then:

- $f(0_E) = 0_F$
  Proof: $f(-u + u) = -f(u) + f(u) \Rightarrow f(0_E) = 0_F$
  All the linear maps $\mathbb{R}^n \to \mathbb{R}^p$ have the form

$$f\left(\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}\right) = \begin{pmatrix} a_{1,1}x_1 + \cdots + a_{1,n}x_n \\ \vdots \\ a_{p,1}x_1 + \cdots + a_{p,n}x_n \end{pmatrix}$$

- $f(-u) = -f(u)$
- If $f: E \to E$, then f is called an endomorphism
- If f is a bijection, it is called an isomorphism

Reminder:

$$f \circ g = 0 \Leftrightarrow Im(g) \subset \mathrm{Ker}(f)$$

Kernel and image

Let $f: E \to F$ be a linear map ($f \in \mathcal{L}(E, F)$)

$$\begin{aligned} Ker(f) &= \{\text{preimages of } 0_F \text{ by } F\} \\ &= \{u \in E \text{ such that } f(u) = 0_F\} \\ &= f^{-1}(\{0_F\}) \end{aligned}$$

$$\begin{aligned} Im(f) &= f(E) \\ &= \{f(u), u \in E\} \\ &= \{v \in F \ \text{ such that } \exists u \in E, v = f(u)\} \end{aligned}$$

Dimension

The dimension of E corresponds to the cardinal of its basis.

$$F \subset G \Rightarrow \dim(F) \leq \dim(G)$$

$$F \subset G \ \text{ and } \ \dim(F) = \dim(G) \Rightarrow F = G$$

Let F and G be two subspaces of E such that $F \cap G = \{0_E\}$, then

$$\dim(F \oplus G) = \dim(F) + \dim(G)$$

Generally, $\dim(F) + \dim(G) = \dim(F + G) + \dim(F \cap G)$

Rank theorem

$f \in \mathcal{L}(E, F)$, E finite dimension, $\dim(E) = \dim(Ker(f)) + \dim(Im(f))$

Matrices

Multiplication

$$\underbrace{A}_{n\times p} \times \underbrace{B}_{p\times q} = \underbrace{C}_{n\times q}$$

e.g. $A = (2 \quad -1)$, $B = \begin{pmatrix} 3 \\ 4 \end{pmatrix}$, $\underbrace{A}_{1\times 2} \times \underbrace{B}_{2\times 1} = \underbrace{(2)}_{1\times 1}$ and $\underbrace{B}_{2\times 1} \times \underbrace{A}_{1\times 2} = \begin{pmatrix} 6 & -3 \\ 8 & -4 \end{pmatrix}$

Matrix of a linear map

Let $f: E \to F$ be a linear map, $\mathcal{B}_1 = (e_1, \dots, e_p)$ a basis of E ($\dim(E) = p$), $\mathcal{B}_2 = (\varepsilon_1, \dots, \varepsilon_n)$ a basis of F ($\dim(F) = n$). In $\mathcal{B}_1$ and $\mathcal{B}_2$.

$$A = Mat(f) = \begin{pmatrix} f(e_1) \text{ coord along } \varepsilon_1 & \cdots & f(e_p) \text{ coord along } \varepsilon_1 \\ \vdots & \cdots & \vdots \\ f(e_1) \text{ coord along } \varepsilon_n & \cdots & f(e_p) \text{ coord along } \varepsilon_n \end{pmatrix}$$

If $u \in E$ has coordinates $X = \begin{pmatrix} x_1 \\ \vdots \\ x_p \end{pmatrix}$ in the basis $\mathcal{B}_1$ and $v = f(u)$ has coordinates $Y = \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix}$ in the basis $\mathcal{B}_2$, then $Y = AX$