SPE Maths Quick Sheet

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Series

- The series of general term u_n, denoted $\sum u_n$, is the sequence of partial sum

$$(S_n)_{n\in\mathbb{N}}, S_n = \sum_{k=0}^{n} u_k$$

- (S_n) converges $\Leftrightarrow \sum u_n$ converges
- Let $\sum u_n$ and $\sum v_n$ be two sequences. Let $\lambda \in \mathbb{R}$
  - $\sum u_n$ CVG and $\sum v_n$ CVG $\Rightarrow \sum(u_n + v_n)$ CVG
  - $(\sum u_n \text{ CVG}) \Rightarrow (\sum \lambda u_n \text{ CVG})$
  - $(\sum \mathrm{u_n} \text{ CVG and } \sum v_n \text{ DVG}) \Rightarrow (\sum u_n + v_n \text{ CVG})$
- **Necessary condition of convergence**

$$\sum u_n \text{ is convergent} \Rightarrow \lim_{n\to+\infty} u_n = 0$$

Series of nonnegative terms

- $\sum u_n$ is a series of nonnegative terms if, $\forall n \in \mathbb{N}, u_n \geq 0$.
- $\sum u_n$ is a series of nonnegative terms, (S_n) the sequence of partial sums: $\sum u_n$ converges $\Leftrightarrow u_n$ is upper-bounded
- Let (u_n) and (v_n) be two sequences such that $\forall n \in \mathbb{N}, 0 \leq u_n \leq v_n$

$$\begin{cases} \sum v_n \text{ CVG} \Rightarrow \sum u_n \text{ CVG} \\ \sum u_n \text{ DVG} \Rightarrow \sum v_n \text{ DVG} \end{cases}$$

- **Geometric series:** $\sum q^n$ CVG $\Leftrightarrow |q| < 1$
- **Riemann series**: $\sum \frac{1}{n^\alpha}, \alpha \in \mathbb{R}$
  - $\sum \frac{1}{n^\alpha}$ converges $\Leftrightarrow \alpha > 1$
- **Criteria of comparison**
  Let (u_n) and (v_n) be two nonnegative real sequences.
  - $u_n = o(v_n) \Rightarrow \begin{cases} \sum v_n \text{ CVG} \Rightarrow \sum u_n \text{ CVG} \\ \sum u_n \text{ DVG} \Rightarrow \sum v_n \text{ DVG} \end{cases}$
  - $u_n \underset{n\to+\infty}{\sim} v_n \Rightarrow \sum u_n$ and $\sum v_n$ are of the same nature
  - $u_n = O(v_n) \Rightarrow$ same as $u_n = o(v_n)$
- *Reminder* **Landau notation**
  - $u_n = o(v_n) \Leftrightarrow \frac{u_n}{v_n} \xrightarrow[n\to+\infty]{} 0$
  - $u_n \underset{n\to+\infty}{\sim} v_n \Leftrightarrow \frac{u_n}{v_n} \xrightarrow[n\to+\infty]{} 1$
  - $u_n = O(v_n) \Leftrightarrow \frac{u_n}{v_n}$ is bounded towards $n \to +\infty$
  - $u_n + o(u_n) \sim u_n$
- Let (u_n) be a real sequence of nonnegative terms. Then

$$\sum(u_n - u_{n-1}) \text{ CVG} \Leftrightarrow (u_n) \text{ CVG}$$

- **Riemann's rule**
  Let (u_n) be a real nonnegative sequence. If $\exists \alpha > 1$ such that $n^\alpha u_n \xrightarrow[n\to+\infty]{} 0$ then $\sum u_n$ CVG
- **D'Alembert/Cauchy test**

$$\text{If either } \begin{cases} \frac{u_{n+1}}{u_n} \to l \\ \sqrt[n]{u_n} \to l \end{cases} \text{ then } \begin{cases} l < 1 \Rightarrow \sum u_n \text{ CVG} \\ l > 1 \Rightarrow \sum u_n \text{ DVG} \\ l = 1 \Rightarrow ? \end{cases}$$

  Using the ratio, it is a D'Alembert test. Using the root, it is a Cauchy test.

Series of arbitrary terms

- **Alternating sequence**
  Let (u_n) be a real sequence. (u_n) is an alternating sequence if there exists a nonnegative sequence (v_n) such that for all $n \in \mathbb{N}$: $u_n = (-1)^n v_n$ or $u_n = (-1)^{n+1} v_n$
- **Alternating series**
  If u_n is an alternating sequence, then the series $\sum u_n$ is called an alternating series.

- Let (u_n) be a real alternating sequence
  (u_n) is decreasing and $\lim_{n\to+\infty} u_n = 0$, then $\sum u_n$ CVG
  $$\left.\begin{array}{c}(|u_n|) \text{ is decreasing} \\ \lim\limits_{n\to+\infty} u_n = 0\end{array}\right\} \Rightarrow \sum u_n \text{ CVG}$$
- We say $\sum u_n$ converges absolutely if the series $\sum |u_n|$ converges.
  - $\sum u_n$ converges absolutely $\Rightarrow \sum u_n$ converges
  - We say $\sum u_n$ is semi-convergent (or conditionally convergent) if it converges but does not converge absolutely.
- Let $\alpha \in \mathbb{R}$. Then $\sum \frac{(-1)^n}{n^\alpha}$ CVG $\Leftrightarrow \alpha > 0$

Generating functions

Ah yes, enslaved probabilities

- **Definition**
  If the possible values for X are $[\![0,n]\!]$, then
  $$G_X(t) = t^0P(X=0) + t^1P(X=1) + \cdots + t^nP(X=N)$$
  $G_X(t)$ is a polynomial
- **General properties**
  - $G_X(1) = 1$
  - $E(X) = G'_X(1)$
  - $Var(X) = G''_X(1) + G'_X(1) - \left(G'_X(1)\right)^2$
- If X and Y are **independent** random variables, then:
  $$G_{X+Y}(t) = G_X(t) \times G_Y(t)$$
- *Reminder* **Bernoulli distribution**
  $$X \rightsquigarrow \text{Bernoulli}(p) \Rightarrow \begin{cases} X \in \{0,1\} \\ P(X=1) = p \\ P(X=0) = 1-p \end{cases}$$
  X has a Bernoulli distribution with parameter p
- *Reminder* **Binomial distribution**
  A binomial distribution is the repetition of a Bernoulli distribution n times. With $i \in [\![1,n]\!]$,
  $$X_i \rightsquigarrow \text{Bernoulli}(p)$$
  $$Y = X_1 + \cdots + X_n \rightsquigarrow \text{B}(n,p)$$
  The probability that there are k X variables equal to 1 is:
  $$P(Y=k) = \binom{n}{k}(1-p)^{n-k}p^k$$
- If X can take an infinite number of values in $\mathbb{N}$,
  $$G_X(t) = \sum_{k=0}^{+\infty} P(X=k)t^k$$

Power series

- **Definition**
  $$\sum a_n x^n = a_0 + a_1x + a_2x^2 + \cdots$$
  - a_n does not depend on x
  - If $x = 0$, all terms of $\sum a_n x^n$ are equal to 0, except when $n = 0$, since $a_0 \times x^0 = a_0 \times 1$
  - If $\sum a_n x^n$ CVG for some values of x, then we have $f(x) = \sum_{n=0}^{+\infty} a_n x^n$
- **Sum and product of power series**
  Let $f(x) = \sum_{n=0}^{+\infty} a_n x^n$ with its radius of convergence R_f and $g(x) = \sum_{n=0}^{+\infty} b_n x^n$ with its radius of convergence R_g, then
  - $f(x) + g(x) = \sum_{n=0}^{+\infty}(a_n + b_n)x^n$
  - $f(x) \times g(x) = \sum_{n=0}^{+\infty} c_n x^n$ where $c_n = \sum_{k=0}^{+\infty} a_k \times b_{n-k}$
- **Radius of convergence**
  Let (a_n) be a real sequence, $\sum a_n x^n$ the P.S. defined by this sequence and the function $f: x \mapsto \sum_{n=0}^{+\infty} a_n x^n$.
  Then $\exists R \in \mathbb{R}_+ \cup \{+\infty\}$ such that
  - $\forall x \in \mathbb{R}, |x| < R$, then $\sum a_n x^n$ CVG ABS
  - $\forall x \in \mathbb{R}, |x| > R$, then $\sum a_n x^n$ DVG

  R is called the radius of convergence of this P.S.
  The set $\{x \in \mathbb{R}, |x| < R\} =]-R, R[$ is called the open disk of convergence of the P.S.
- **Determining the radius of convergence** of a power series
  Ratio test (D'Alembert's rule)
  Let $\sum a_n x^n$ be a power series. If $\left|\frac{a_{n+1}}{a_n}\right| \to l$, then $R = \frac{1}{l}$
  (considering that $\frac{1}{0} = +\infty$ and $\frac{1}{+\infty} = 0$)
- **Power series of basic functions**
  - e^x
  $$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$
  $$= \sum_{n=0}^{+\infty} \frac{x^n}{n!}$$

$$R = +\infty$$

- $\ln(1+x)$

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots$$
$$= \sum_{n=1}^{+\infty} (-1)^{n+1} \frac{x^n}{n}$$
$$R = 1$$
$$\ln(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \cdots$$
$$= -\sum_{n=1}^{+\infty} \frac{x^n}{n}$$

- $(1+x)^\alpha$

$$(1+x)^\alpha = \sum_{n=0}^{+\infty} a_n x^n$$
$$a_n = \frac{\alpha(\alpha-1)\dots(\alpha-n)}{n!}$$
$$R = \begin{cases} 1 \text{ if } \alpha \notin \mathbb{N} \\ +\infty \text{ if } \alpha \in \mathbb{N} \end{cases}$$

- $\sin(x)$

$$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots$$
$$= \sum_{n=0}^{+\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$$
$$R = +\infty$$

- $\cos(x)$

$$\cos(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots$$
$$= \sum_{n=0}^{+\infty} (-1)^n \frac{x^{2n}}{(2n)!}$$
$$R = +\infty$$

Infinite discrete random variable

- **Definition**

  Let $(\Omega, P, \mathcal{P}(\Omega))$ be a probability space, and $x: \Omega \to \mathbb{R}$ our random variable. X is a discrete infinite random variable if the values of $X(\Omega)$ are indexable by $\mathbb{N}$:

  $$X(\Omega) = \{x_0, x_1, x_2, \dots\} = \{x_k, k \in \mathbb{N}\}$$

  (We will study IDRVs where $X(\Omega) \subset \mathbb{N}$, so X is an Infinite Integer Random variable)

  - $P(\Omega) = 1 \Leftrightarrow \sum_{n=0}^{+\infty} P(X=n) = 1$

    (The series of general term $P(X=n)$ is equal to 1)
  - $P(X \in A) = \sum_{n \in A} P(X=n)$

- **Geometric distribution**

  Let $p \in]0,1[$ and $X \rightsquigarrow \text{Bernoulli}(p)$. Let Y be the number of tries needed to get the first $X = 1$, with each try being independent.

  $$\forall n \in \mathbb{N}^*, P(Y=n) = (1-p)^{n-1} \times p$$

  Y is a geometric distribution R.V. $\Leftrightarrow Y \rightsquigarrow \mathcal{G}(p)$

- **Expected value and variance**

  $$E(X) = \sum_{n \in X(\Omega)} n \times P(X=n)$$
  $$V(X) = \sum_{n \in X(\Omega)} \left(n - E(X)\right)^2 P(X=n)$$
  $$\sigma(X) = \sqrt{V(X)}$$

  - If the sum of power series in $E(X)$ diverges, X has no expected value or variance
  - If the sum of power series in $E(X)$ converges but the one in $V(X)$ diverges, X has an expected value but no variance

- **Generating function**

  $$G_X(t) = P(X=0)t^0 + P(X=1)t^1 + \cdots + P(X=n)t^n + \cdots$$
  $$= \sum_{n=0}^{+\infty} P(X=n)t^n$$

- The convergence radius of the resulting series is ≥ 1
- G_X exists and is continuous over at least $[-1, 1]$, and $G_X(1) = 1$
- G_X is C^∞ over $]-1, 1[$
  - *Reminder*
    f is C^0 over I means that it is continuous over I
    f is C^1 over I means that it is differentiable and that f' is continuous over I
    f is C^n over I means that it is differentiable and that $f^{(n+1)}$ is continuous
- If X and Y are independent IIRV, $G_{X+Y} = G_X + G_Y$
- If X has an expected value ($\Leftrightarrow \sum_{n \in X(\Omega)} n \times P(X = n)$ converges), then G_X is differentiable for $t = 1$ and $E(X) = G_X'(1)$
- If X has a variance, G_X' is differentiable for $t = 1$ and $V(X) = G_X''(1) + G_X'(1) - \left(G_X'(1)\right)^2$

Linear algebra, Second Edition

Vector space

- Let E be a $\mathbb{K}$ vector space. Then $\forall(\lambda,\mu)\in\mathbb{K}^2, \forall(x,y)\in E^2$,
  - $(\lambda+\mu)x=\lambda x+\mu x$
  - $\lambda(\mu x)=(\lambda\mu)x$
  - $\lambda(x+y)=\lambda x+\lambda y$
  - $1x=x$

Vector subspace

- **Definition**
  Let E be a $\mathbb{K}$-vs. F is a vector subspace of E if:
  - $F\subset E$
  - $F\neq\emptyset$
  - $\forall(u,v)\in F^2, \forall(\alpha,\beta)\in\mathbb{K}^2, (\alpha u+\beta v)\in F$ (Closure)
- **Operations**
  Let E be a $\mathbb{K}$-vs, F and G be two vector subspaces of E
  - $F\cap G$ is a vector subspace of E
  - $F+G=\{z\in E, \exists(x,y)\in F\times G, z=x+y\}$ is a vector subspace of E
- **Direct sum**
  Let E be a $\mathbb{K}$-vs, F and G be two vector subspaces of E. We say that F and G are in direct sum if
  $$\forall(x,y)\in F\times G, x+y=0_E\Rightarrow x=y=0_E$$
- The following are all equivalent:
  - F and G are in direct sum
  - $F\cap G=\{0_E\}$
  - $\forall z\in E, \exists!(x,y)\in F\times G, z=x+y$
- **Supplementary subspaces**
  F and G are supplementary in E (written $E=F\oplus G$) if:
  - $F\cap G=\{0_E\}$
  - $E=F+G$

  Saying that two subspaces are in direct sum or that they are supplementary is equivalent

Spanned vector subspaces

- **Definition**
  Let E be a $\mathbb{K}$-vs and $X\subset E$. There exists a subspace of E containing X. It is called the vector subspace of E spanned by X, denoted $\mathrm{Span}(X)$. We say that X spans G or that G is spanned by X.
  With $X=\{x_1,\dots,x_n\}\subset E$ and $(\lambda_1,\dots,\lambda_n)\in\mathbb{K}^n$:
  $$\mathrm{Span}(X)=\{\lambda_1x_1+\lambda_2x_2+\dots+\lambda_nx_n\}$$

Linear map

- Let E and F be two $\mathbb{K}$-vs and $f: E\to F$. We say that f is a linear map if, denoted $f\in\mathcal{L}(E,F)$:
  - $\forall(u,v)\in E^2, f(u+v)=f(u)+f(v)$
  - $\forall u\in E, \forall\lambda\in\mathbb{K}, f(\lambda u)=\lambda f(u)$
  - Or, with just one definition:
    $$\forall(u,v)\in E^2, \forall\lambda\in\mathbb{K}, f(\lambda u+v)=\lambda f(u)+f(v)$$
- **Kernel, image**
  Let $f\in\mathcal{L}(E,F)$.
  - $Ker(f)=\{x\in E, f(x)=0_F\}$
  - $Im(f)=\{f(x), x\in E\}=\{y\in F, \exists x\in E, y=f(x)\}$
- **Properties**
  Let $f\in\mathcal{L}(E,F)$
  - $Ker(f)$ and $Im(f)$ are vector subspaces (of E and F respectively)
  - $Ker(f)=\{0_E\}\Leftrightarrow f$ is injective
  - $Im(f)=F\Leftrightarrow f$ is surjective

Basis and dimension

- **Linearly independent set**
  Let E be a $\mathbb{K}$-vs, and $L=\{x_1,\dots,x_n\}\subset E$. We say that L is a linearly independent set if:

$$\forall(\lambda_1, \dots, \lambda_n) \in \mathbb{K}^n, \sum_{i=1}^{n} \lambda_i x_i = 0_E \Rightarrow \lambda_1 = \dots = \lambda_n = 0_{\mathbb{K}}$$

- **Basis**
  Let E be a $\mathbb{K}$-vs and $B = \{x_1, \dots, x_n\} \subset E$. B is a basis of E means that B is linearly independent and E is spanned by B.
- **Finite vector space**
  Let E be a $\mathbb{K}$-vs. E is a finite vector space if E is spanned by a finite set of vectors of E.
  e.g.: $\mathbb{R}_n[X]$ is spanned by $B = \underbrace{(1, X, X^2, \dots, X^n)}_{\text{finite set of } n+1 \text{ vectors}}$: it is a finite vector space
- **Dimension**
  Let E be a $\mathbb{K}$-vs of finite dimension. All the basis of E have the same cardinality, which is called the dimension of E and is written $\dim(E)$
- **Incomplete basis theorem**
  If $\dim(E) = n$
  - A set S which is independent has at most n vectors. If it is not a spanning set, then
    - $Card(S) < n$
    - We can add vectors until we get a basis
  - A spanning set S has at least n vectors. If it is not independent:
    - $Card(S) > n$

  We can remove vectors until we get a basis
- **Propositions**
  Let E be a $\mathbb{K}$-vs of finite dimension.
  - If $E \neq \{0\}$, then E admits at least one basis
  - Let F be a vector subspace of E. Then F admits at least one supplementary subspace in E
  - Let F and G be two supplementary subspaces in E, B as basis of F and B' a basis of G. Then $B \cup B'$ is a basis of E
  - $E = F \oplus G \Rightarrow \dim E = \dim F + \dim G$
  - Let F and G be two vector subspaces of E such that
    $$\left.\begin{array}{c} F \subset G \\ \dim F = \dim G \end{array}\right\} \Rightarrow F = G$$
  - Let $f \in \mathcal{L}(E, F)$. If f is bijective, then $\dim E = \dim F$
  - Let F and G be two vector subspaces of E:
    $$\dim F + G = \dim F + \dim G - \dim F \cap G$$
- **Rank theorem (rank-nullity theorem)**
  Let $f \in \mathcal{L}(E, F)$ where E and F are vector spaces of finite dimensions.
  $$\dim E = \dim Ker(f) + \dim Im(f)$$

Matrix representation of a linear map

Let $f: E \to F$ be a linear map, $\mathcal{B}_1 = (e_1, \dots, e_p)$ a basis of E ($\dim(E) = p$), $\mathcal{B}_2 = (\varepsilon_1, \dots, \varepsilon_n)$ a basis of F ($\dim(F) = n$). In $\mathcal{B}_1$ and $\mathcal{B}_2$.

$$A = Mat(f) = \begin{pmatrix} f(e_1) \text{ coord along } \varepsilon_1 & \cdots & f(e_p) \text{ coord along } \varepsilon_1 \\ \vdots & \cdots & \vdots \\ f(e_1) \text{ coord along } \varepsilon_n & \cdots & f(e_p) \text{ coord along } \varepsilon_n \end{pmatrix}$$

If $u \in E$ has coordinates $X = \begin{pmatrix} x_1 \\ \vdots \\ x_p \end{pmatrix}$ in the basis $\mathcal{B}_1$ and $v = f(u)$ has coordinates $Y = \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix}$ in the basis $\mathcal{B}_2$, then $Y = AX$